Statistical Data Analysis for Physicists
Homework 1

Artur Palha nº46724

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Exercise 1. Some basics.

(a) 1. If $A \subseteq B$ then $B = (B \cap \bar{A}) \cup A$. From this follows that:

$$P(B) = P\left((B \cap \bar{A}) \cup A\right)$$

since $B \cap \bar{A}$ and A are disjoint we can use the kolmogorov axioms and obtain:

$$P(B) = P\left(B \cap \bar{A}\right) + P(A)$$

which implies that:

$$P(B) \geq P(A)$$

because, using the kolmogorov axioms:

$$P\left(B \cap \bar{A}\right) \geq 0 \quad \blacksquare$$

2. We wish to prove that:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

We first note that $A \cap \bar{B}$ and B are disjoint sets whose union is $A \cup B$. Hence, by kolmogorov axioms follows that:

$$P(A \cup B) = P\left(A \cap \bar{B}\right) + P(B) \tag{1}$$

Also, $A \cap \bar{B}$ and $A \cap B$ are disjoint sets whose union is A, so kolmogorov axioms gives:

$$P(A) = P\left(A \cap \bar{B}\right) + P(A \cap B)$$

Solving for $P\left(A \cap \bar{B}\right)$ and replacing in Equation (1) gives:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \blacksquare$$

3. We wish to prove that $P(A|B)$ satisfies the Kolmogorov axioms. That means:

$$\begin{cases} 0 \leq P(A|B) \leq 1 \\ P((A \cup C)|B) = P(A|B) + P(A|C) \quad \text{if } A \cap C = \emptyset \\ P(B|B) = 1 \end{cases}$$

I. Lets show that it satisfies the first axiom.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \geq 0, \qquad \text{since } P(B) \geq 0 \text{and } P(A \cap B) \geq 0.$$

Now we have to prove that:

$$P(A|B) \leq 1$$

Which is to prove that:

$$P(A \cap B) \leq P(B)$$

This follows directly from what has been proven in the first exercise, because $A \cap B \subseteq B$.

II. We will now show that it satisfies the second axiom.

$$P((A \cup C)|B) = P(A|B) + P(C|B), \qquad \text{since } A \cap C = \emptyset$$

By definition:

$$P(A|B) + P(C|B) = \frac{P(A \cap B)}{P(B)} + \frac{P(A \cap B)}{P(B)}$$

Since $A \cap B$ and $C \cap B$ are disjoint we can use Kolmogorov axioms to obtain:

$$P(A|B) + P(C|B) = \frac{1}{P(B)} \left(P((A \cap B) \cup (C \cap B))\right)$$

It's known from set theory that $(A \cap B) \cup (C \cap B) = (A \cup C) \cap B$. That means:

$$P(A|B) + P(C|B) = \frac{1}{P(B)} \left(P((A \cup C) \cap B)\right).$$

Which is just, by definition, $P((A \cup C)|B)$.

III. Finally the third axiom. By definition:

$$P(B|B) = \frac{P(B \cap B)}{P(B)} = 1. \quad \blacksquare$$

(b) 1. We just start from one and try to get to the other. Let's start with the second one:

$$P(A|B) = P\left(A|\bar{B}\right)$$

By definition:

$$\frac{P(A \cap B)}{P(B)} = \frac{P\left(A \cap \bar{B}\right)}{P\left(\bar{B}\right)}$$

We know that $A \cap B$ and $A \cap \bar{B}$ are disjoint and their union is A, so:

$$\frac{P(A \cap B)}{P(B)} = \frac{P(A) - P(A \cap B)}{P\left(\bar{B}\right)}$$

Using the property that $P\left(\bar{B}\right) = 1 - P(B)$, we obtain:

$$\frac{P(A \cap B)}{P(B)} = P(A)$$

The term on the left side is just, by definition:

$$P(A|B) = P(A)$$

Using the definition, we find that this implies to say that:

$$P(A|B) = P(A)\, P(B)$$

Which is the first equation. In this deduction we just need to insure that

$$P(A), P(B) \neq 0, 1. \quad \blacksquare$$

2. We suppose that:
$$P\left(\bar{A}\cap\bar{B}\right)=P\left(\bar{A}\right)P\left(\bar{B}\right) \tag{2}$$
is true. If, in the end, we obtain a true proposition, that means Equation (2) is true.
$$\begin{aligned} P\left(\bar{A}\cap\bar{B}\right) &= P\left(\bar{A}\right)P\left(\bar{B}\right) \\ &= \left(1-P\left(A\right)\right)\left(1-P\left(B\right)\right) \\ &= 1-P\left(B\right)-P\left(A\right)+P\left(A\right)P\left(B\right) \end{aligned}$$
Now, since A and B are independent, which means that: $P\left(A\right)P\left(B\right)=P\left(A\cap B\right)$, follows that:
$$P\left(\bar{A}\cap\bar{B}\right)=1-\left(P\left(B\right)+P\left(A\right)-P\left(A\cap B\right)\right)$$
Now using what has been demonstrated on Exercise (a2), follows that:
$$P\left(\bar{A}\cap\bar{B}\right)=1-P\left(A\cup B\right)$$
Which is:
$$P\left(\bar{A}\cap\bar{B}\right)+P\left(A\cup B\right)=1$$
Because $\bar{A}\cap\bar{B}$ and $A\cup B$ are disjoint, we can use kolmogorov axioms and obtain:
$$P\left(\left(\bar{A}\cap\bar{B}\right)\cup\left(A\cup B\right)\right)=1$$
But we know that $\bar{A}\cap\bar{B})\cup(A\cup B)=\Omega$. Then we obtain:
$$P\left(\Omega\right)=1$$
Which is true. Then Equation (2) is true. ■

Exercise 2. Photo-tube

We use the following model. The probability that an event A occurs in the interval $[t,t+h]$ is supposed to be $\lambda h+\mathcal{O}(h)$, where λ is the intensity. λ is the number of particles by unity of time. $P\left(t\right)$ is the probability that, starting in T, no event occurs before t. Then, the probability that no event occurs before $t+h$ is just the product of the probability that no event has occurred before t and the probability that no event has occurred between t and $t+h$, that is:
$$P\left(t+h\right)=P\left(t\right)\left(1-\lambda h+\mathcal{O}(h)\right)$$
if we subtract $P\left(t\right)$ on both sides of the equation and divide by h and take the limit $h\to 0$, we get:
$$\frac{\mathrm{d}P\left(t\right)}{\mathrm{d}t}=-\lambda P\left(t\right)$$
Solving this differential equation we get:
$$P\left(t\right)=P\left(0\right)e^{-\lambda t}$$
Using the normalization condition:
$$\int P\left(t\right)\mathrm{d}t=1$$
We get that $P\left(0\right)=\lambda$, which means that:
$$P\left(t\right)=\lambda e^{-\lambda t}. \quad ■$$

Exercise 3. Cosmic ray muons

1. 1.1 Computation of A

   If $f(x;\lambda)$ is a probability, then:

$$\int_0^{+\infty} f(x;\lambda)\,\mathrm{d}x = 1$$

   Which means that:

$$\int_0^{+\infty} Ae^{-\frac{x}{\lambda}}\,\mathrm{d}x = 1$$

   Integration gives:

$$\begin{aligned} A\left[-\lambda e^{-\frac{x}{\lambda}}\right]\Big|_0^{+\infty} &= 1 \\ A\lambda &= 1 \\ A &= \frac{1}{\lambda}. \quad \blacksquare \end{aligned}$$

   1.2 Cumulative distribution of the law

   We wish to find the cumulative distribution, which is, by definition:

$$F(x;\lambda) = \int_0^{x} Ae^{-\frac{x'}{\lambda}}\,\mathrm{d}x'$$

   Integrating:

$$\begin{aligned} F(x;\lambda) &= \frac{1}{\lambda}\left[-\lambda e^{-\frac{x'}{\lambda}}\right]\Big|_0^{x} \\ &= 1 - f(x;\lambda). \quad \blacksquare \end{aligned}$$

2. 2.1 Derivation of the mean μ

   By definition of the mean μ:

$$\mu = \int_0^{+\infty} x f(x;\lambda)\,\mathrm{d}x$$

   Integration by parts gives:

$$\mu = \frac{1}{\lambda}\left(\left[-\lambda x e^{-\frac{x}{\lambda}}\right]\Big|_0^{+\infty} + \int_0^{+\infty} \lambda e^{-\frac{x}{\lambda}}\,\mathrm{d}x\right)$$

   Easily we see that the first term on the right side of the last equation is zero, then:

$$\mu = \int_0^{+\infty} e^{-\frac{x}{\lambda}}\,\mathrm{d}x$$

   Which gives directly:

$$\mu = \lambda. \quad \blacksquare$$

   2.2 Derivation of the standard deviation σ

   By definition of the standard deviation:

$$\begin{aligned} \sigma^2 &= \mathrm{Var}(x) \\ &= \int_0^{+\infty} (x-\mu)^2 f(x;\lambda)\,\mathrm{d}x \end{aligned}$$

   Computation of σ^2:

$$\sigma^2 = \int_0^{+\infty} \frac{(x-\lambda)^2}{\lambda} e^{-\frac{x}{\lambda}}\,\mathrm{d}x$$

Expanding the square:

$$\sigma^2 = \int_0^{+\infty} \frac{x^2}{\lambda} e^{-\frac{x}{\lambda}} \mathrm{d}x + \int_0^{+\infty} \frac{\lambda^2}{\lambda} e^{-\frac{x}{\lambda}} \mathrm{d}x - \int_0^{+\infty} \frac{2x\lambda}{\lambda} e^{-\frac{x}{\lambda}} \mathrm{d}x$$

The only term we have to determine is the first one, the other two we have computed already. The second gives λ^2 and the third gives $-2\lambda^2$. Therefore we have:

$$\sigma^2 = \int_0^{+\infty} \frac{x^2}{\lambda} e^{-\frac{x}{\lambda}} \mathrm{d}x - \lambda^2$$

Integration by parts gives:

$$\sigma^2 = \frac{1}{\lambda} \left(\left[-x^2 \lambda e^{-\frac{x}{\lambda}} \right] \Big|_0^{+\infty} + 2\lambda^2 \int_0^{+\infty} \frac{x}{\lambda} e^{-\frac{x}{\lambda}} \mathrm{d}x \right) - \lambda^2$$

The first term on the right side of the last equation is equal to zero and the second term is just $2\lambda^2$, which gives:

$$\sigma^2 = 2\lambda^2 - \lambda^2 = \lambda^2$$

That gives directly:

$$\sigma = \lambda. \quad \blacksquare$$

3. We want to prove that:

$$P\left((x \in [x_0, x_0 + x']) \,|\, (x > x_0) \right) = P\left(x < x'. \right)$$

By definition of conditional probability we have:

$$P\left((x \in [x_0, x_0 + x']) \,|\, (x > x_0) \right) = \frac{P\left(x \in [x_0, x_0 + x'] \right)}{P\left(x > x_0 \right)}$$

Lets calculate things separately:

3.1 Calculation of $P\left(x > x_0\right)$

$$\begin{aligned} P\left(x > x_0\right) &= 1 - P\left(x < x_0\right) \\ &= 1 - \frac{1}{\lambda} \int_0^{x_0} e^{-\frac{x}{\lambda}} \mathrm{d}x \end{aligned}$$

Which, by integration gives:

$$P\left(x > x_0\right) = e^{-\frac{x_0}{\lambda}}$$

3.2 Calculation of $P\left(x \in [x_0, x_0 + x']\right)$ It's known that:

$$P\left(x \in [x_0, x_0 + x']\right) = P\left(x < x_0 + x'\right) - P\left(x < x_0\right)$$

Which is:

$$P\left(x \in [x_0, x_0 + x']\right) = \frac{1}{\lambda} \int_0^{x_0 + x'} e^{-\frac{x}{\lambda}} \mathrm{d}x - \frac{1}{\lambda} \int_0^{x_0} e^{-\frac{x}{\lambda}} \mathrm{d}x$$

Integrating:

$$\begin{aligned} P\left(x \in [x_0, x_0 + x']\right) &= e^{-\frac{x_0}{\lambda}} - e^{-\frac{x_0 + x'}{\lambda}} \\ &= e^{-\frac{x_0}{\lambda}} \left(1 - e^{-\frac{x'}{\lambda}} \right) \end{aligned}$$

Using the preceding calculations we see that:

$$\begin{aligned} P\left(x \in [x_0, x_0 + x']\right) &= \frac{e^{-\frac{x_0}{\lambda}} \left(1 - e^{-\frac{x'}{\lambda}} \right)}{e^{-\frac{x_0}{\lambda}}} \\ &= 1 - e^{-\frac{x'}{\lambda}} \end{aligned}$$

Having in mind the results obtained previously, we can easily see that:

$$1 - e^{-\frac{x'}{\lambda}} = P\left(x < x'\right)$$

Which ends our demonstration. ■

4. The time that the muon lived prior to entering the detector doesn't play a role in determining the mean life time because it follows an exponential distribution. Lets be more precise. What we are stating here, in mathematical language is just:

$$P\left(\left(t \in [t_0, t_0 + t']\right) \mid (t > t_0)\right) = P\left(t < t'\right)$$

Which is what we have just proved in the previous question. ■

Exercise 4. Addition of random variables

1. Using the general expression for the distribution of the sum of two random variables.

We know that given two random variables X and Y with joint density $f(x, y)$, we have:

$$f_{X+Y}(u) = \int_{-\infty}^{+\infty} f\left(x, u - x\right) \mathrm{d}x$$

Given that $u = x + y$. Since we are dealing with independent variables, we have:

$$f\left(x, y\right) = f_X(x) f_Y(y)$$

Meaning that we have:

$$f_{X+Y}(u) = \int_{-\infty}^{+\infty} f_X\left(x\right) f_Y\left(u - x\right) \mathrm{d}x$$

For discrete random variables we have:

$$f_{N+M}(u) = \sum_n f_N\left(n\right) f_M\left(u - n\right)$$

If we use this to our two discrete random variables that follow two Poisson distributions of parameter μ_N and μ_M we obtain:

$$f_{N+M}(u) = \sum_n \frac{\mu_N^n e^{-\mu_N}}{n!} \frac{\mu_M^{u-n} e^{-\mu_M}}{(u-n)!}$$

Which is just a Poisson distribution with parameter $(\mu_N + \mu_M)$. ■

2. Employing the corresponding relation for the generating function.

By definition, a generating function of probability density function of a discrete random variable X is just:

$$G_X(s) = \sum_{r=0}^{\infty} s^r f_X\left(r\right)$$

For a Poisson distribution of parameter μ we will have:

$$\begin{aligned} G_X(s) &= \sum_{r=0}^{\infty} (\mu s)^r \frac{e^{-\mu}}{r!} \\ &= e^{-\mu} \sum_{r=0}^{\infty} \frac{(\mu s)^r}{r!} \\ &= e^{\mu(s-1)} \end{aligned}$$

Now, to solve this problem we just use the fact that the generating function of the sum $Z = N + M$ is just:

$$G_Z(s) = G_N(s)G_M(s)$$

Because, using the convolution theorem, we know that the fourier transform of the convolution of two functions is the product of the fourier transforms of the functions. Hence the previous result. This is just:

$$G_Z(s) = e^{(\mu_N+\mu_M)(s-1)}$$

This is just the generating function of a Poisson distribution with parameter $(\mu_N+\mu_M)$. Hence $Z \sim Poisson(\mu_N + \mu_M)$. ■

Exercise 5. Generating function for a discrete random variable.

1. 1.a We wish to show that:

$$G(z) \equiv \sum_{k=0}^{\infty} z^k p(k)$$

is defined for all z satisfying $|z| \leq 1$. If $|z| \leq 1$ then $z = ae^{i\theta}$ with $a \in [0,1]$.

$$\therefore \quad G(z) = \sum_{k=0}^{\infty} a^k e^{ik\theta} p(k)$$

If this series is absolutely convergent it means that converges. So we will check for the convergence of $|G(z)|$.

$$|G(z)| = \sum_{k=0}^{\infty} a^k p(k)$$

We easily see that:

$$|G(z)| \leq \sum_{k=0}^{\infty} a^k$$

because $p(k) \leq 1$. Therefore, if This last series converges, we assure convergence of $|G(z)|$which, in turn, assures convergence of $G(z)$. The series:

$$\sum_{k=0}^{\infty} a^k$$

is just a geometric series that converges to $\frac{1}{1-a}$ when $a < 1$. When $a = 1$ is even easier. We just notice that:

$$|G(z)| = \sum_{k=0}^{\infty} p(k)$$

Which, since $p(k)$ is a probability function, is just equal to unity. Therefore we have just proved that the series converges for all $z \leq 1$. ■

1.b We wish to prove that the moments (provided they exist) are given by:

$$\langle K^n \rangle = \left(z\frac{\mathrm{d}}{\mathrm{d}z} \right)^n G(z) \bigg|_{z=0}$$

We notice that $Z = e^{i\theta}$. Then, it follows:

$$G(Z) = G\left(e^{i\theta}\right) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(i\theta k)^n}{n!} p(k)$$

Where we expanded the exponential in series about 0. Rearranging:

$$\begin{aligned} G\left(e^{i\theta}\right) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(i\theta k)^n}{n!}p\left(k\right) \\ &= \sum_{n=0}^{\infty}\frac{\langle(i\theta k)^n\rangle}{n!} \\ &= \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!}\langle K^n\rangle \end{aligned}$$

Where we have used the definition of the mean $\langle K^n\rangle$. Direct inspection of this last equation leads to:

$$i^k\langle K^n\rangle = \left.\frac{\mathrm{d}^n G\left(e^{i\theta}\right)}{\mathrm{d}\theta^n}\right|_{\theta=0} \tag{3}$$

We just have to make a bridge between this and the equation we want. We just have to notice that, since $Z = e^{i\theta}$:

$$\frac{\mathrm{d}G\left(Z\right)}{\mathrm{d}\theta} = \frac{\mathrm{d}G}{\mathrm{d}Z}\frac{\mathrm{d}Z}{\mathrm{d}\theta} = \frac{\mathrm{d}G}{\mathrm{d}Z}iZ$$

by the chain rule. Therefore:

$$\frac{\mathrm{d}^n G\left(Z\right)}{\mathrm{d}\theta^n} = \left.\left((iZ)\frac{\mathrm{d}G}{\mathrm{d}Z}\right)^n\right|_{Z=1}$$

Combining this with Equation (3), we obtain the initial relation, having in mind that $Z = 1 \Rightarrow \theta = 0$ ■

2. 2.a We wish to determine the generating function of the Poisson distribution

   We have done this on Exercise 4, therefore it's useless to repeat that. We just present the result:

$$G_X(s) = e^{\mu(s-1)}$$

   2.b We wish to derive from the generating function of the Poisson distribution the relation:

$$\langle K(K-1)(K-2)\ldots(K-n+1)\rangle = \lambda^n.$$

   We know that the generating function is:

$$G_K(z) = \sum_{k=0}^{\infty} z^k P\left(k\right)$$

   Which is just $\langle z^k\rangle$. Differentiating with respect to z yelds:

$$\frac{\mathrm{d}G(z)}{\mathrm{d}z} = \sum_{k=0}^{\infty} kz^{k-1}P\left(k\right)$$

   We easily realize that:

$$\frac{\mathrm{d}^n G(z)}{\mathrm{d}z^n} = \sum_{k=0}^{\infty} k(k-1)(k-2)\ldots(k-n+1)z^{k-n}P\left(k\right) = \langle k(k-1)(k-2)\ldots(k-n+1)z^{k-n}\rangle$$

   It's easy to see that:

$$\left.\frac{\mathrm{d}^n G(z)}{\mathrm{d}z^n}\right|_{z=1} = \langle k(k-1)(k-2)\ldots(k-n+1)\rangle$$

Differentiating the expression we have for the generating function:

$$\left.\frac{\mathrm{d}^n e^{\lambda(z-1)}}{\mathrm{d}z^n}\right|_{z=1} = \lambda^n$$

This ends our deduction. ■

3. 3.a We want to show that the mean of the Poisson distribution $p(\lambda; k)$ is given by $\langle K\rangle = \lambda$. When $n = 1$ by direct substitution we obtain:

$$\langle K\rangle = \lambda. \quad \blacksquare$$

3.b We want to show that the variance of the Poisson distribution $p(\lambda; k)$ is given by $\mathrm{Var}(K) = \lambda$. We now that:

$$\mathrm{Var}(K) = \langle (K-\lambda)^2\rangle.$$

Expanding we obtain:

$$\begin{aligned} \mathrm{Var}(K) &= \langle K^2 - 2K\lambda + \lambda^2\rangle \\ &= \langle K^2\rangle - 2\lambda^2 + \lambda^2 \\ &= \langle K^2\rangle - \lambda^2 \end{aligned} \tag{4}$$

To finish we just have to notice that:

$$\langle K(K-1)\rangle = \lambda^2$$

Expanding and using the linearity of the expected value:

$$\langle K^2\rangle - \langle K\rangle = \lambda^2$$

Which, rearranging and substituting on Equation (4) leads to:

$$\mathrm{Var}(K) = \lambda^2 + \lambda - \lambda^2 = \lambda. \quad \blacksquare$$